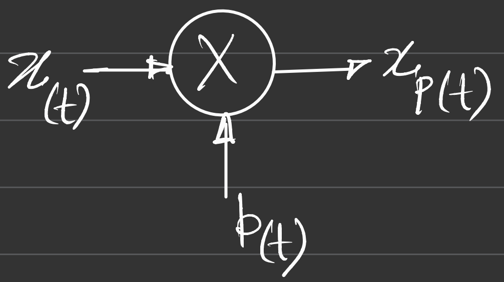
Signal Analysis & Communication ECE 355

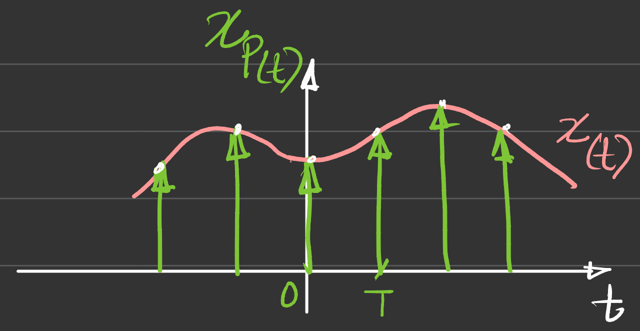
Ch. 7.2 : RECONSTRUCTION FROM SAMPLES

Lecture 29

22-11-2023

Recall: Impulse Train Sampling

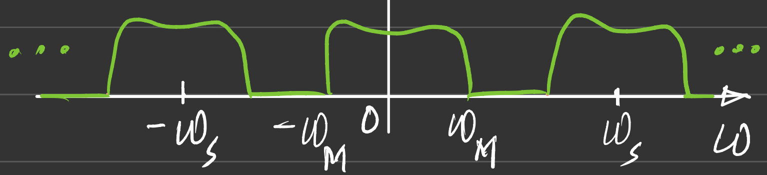




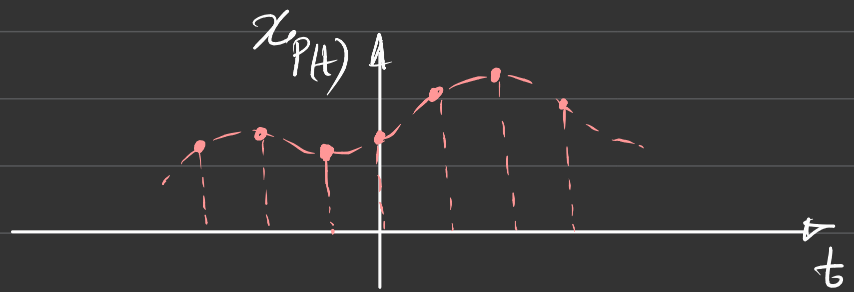
$$\omega_s = \frac{2\pi}{T}$$

$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$\omega_s > 2\omega_M$
$\Rightarrow$ No overlapping.



Ch. 7.2. RECONSTRUCTION FROM SAMPLES $x(nT)$



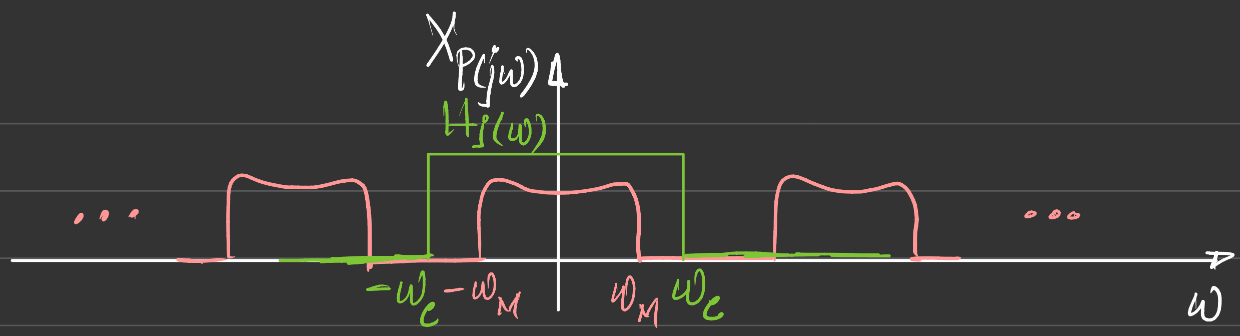
Interpolation

- Determination of "missing" part of a sig. b/w known parts.
- Fitting of a CT sig. to a set of sample values.

Ideal Interpolation with 'sinc function

- For a band-limited sig. (limited by ω_M), if the sampling instants are sufficiently closed ($T\downarrow$, $f_s\uparrow = 1/T\downarrow$, $f_s \geq 2f_m$), then the sig. can be reconstructed exactly.

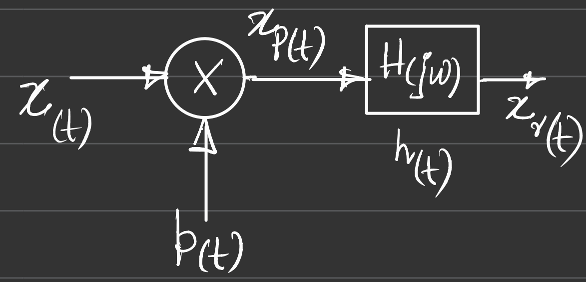
- That is, through the use of a low-pass filter, exact interpolation can be carried out b/w the sampling points.



$$H_{(j\omega)} = \begin{cases} T, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

- To understand the reconstruction of $x_{(t)}$ as a process of interpolation, lets look at the effect of LPF in time-domain
- The o/p of the filter is:

$$X_{r(j\omega)} = X_{p(j\omega)} H_{(j\omega)}$$



$$x_{(t)} = x_{r(t)} = x_{p(t)} * h_{(t)}$$

$$= \left(\sum_{n=-\infty}^{\infty} x_{(nT)} \underline{\delta_{(t-nT)} \right) * h_{(t)}}$$

$$= \sum_{n=-\infty}^{\infty} x_{(nT)} \underline{h_{(t-nT)}} \quad - ①$$

- Above eqn. describes how to fit a continuous curve b/w the sample points $x_{(nT)}$ & consequently represents an interpolation formula.
- For ideal lowpass filter, represented by $H_{(j\omega)}$, the impulse response, $h_{(t)}$, is given as:

$$H_{(j\omega)} = \begin{cases} T & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

$$\Updownarrow$$

$$h_{(t)} = \frac{T \sin(\omega_c t)}{\pi t}$$

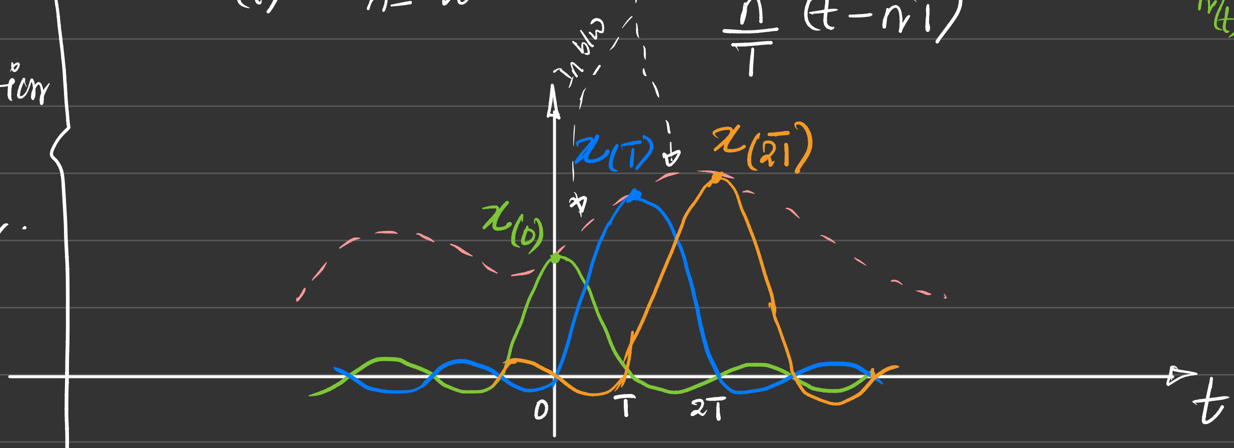
- Usually, we use $\omega_c = \frac{\omega_s}{2} = \frac{2\pi/T}{2} = \frac{\pi}{T}$

- Accordingly,

$$x_{r(t)} = \sum_{n=-\infty}^{\infty} x_{(nT)} \, \frac{\sin\left(\frac{\pi}{T}(t-nT)\right)}{\frac{\pi}{T}(t-nT)} \quad \text{—} (2)$$

Inserting $h_{(t)}$ in eqn (1)

Superposition of sinc function.



- That's called "band-limited interpolation" [if $x_{(t)}$ is band limited]

- In many cases, it is preferable to use a less accurate but simpler filter, or equivalently, a simpler interpolating function than the function in eqn (2).

- Interpolation with Zero-order Hold:

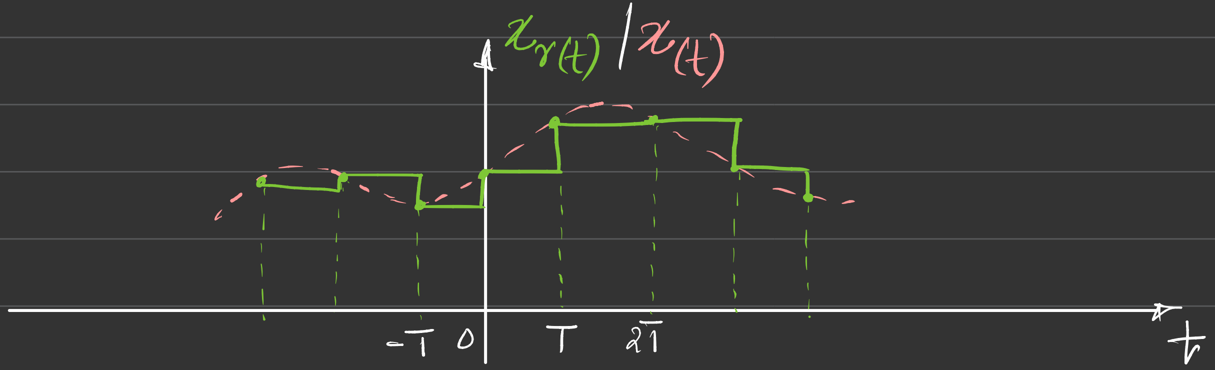
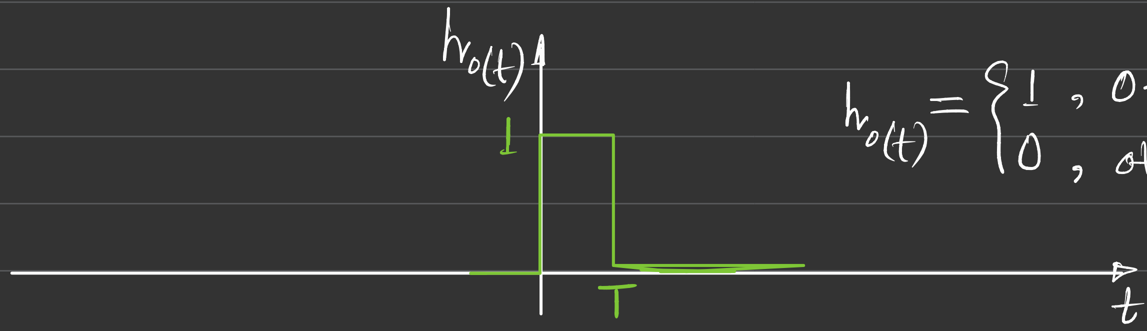


Fig. 1

$$x_{r(t)} = x_{(kT)} \quad \text{for} \quad kT \leq t < (k+1)T$$

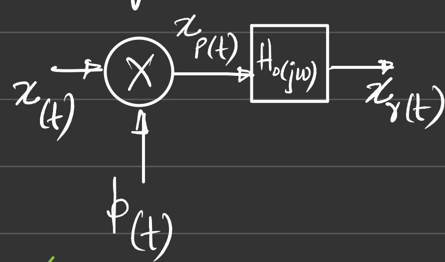
- "Zero-order hold" can be viewed as a form of interpolation b/w sample values in which the interpolating function $h(t)$ is the impulse response $h_o(t)$ given as:



$$h_o(t) = \begin{cases} 1, & 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

- $x_r(t)$ can mathematically be represented using $h_o(t)$ as:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\, h_o(t-nT)$$



$$= \sum_{n=-\infty}^{\infty} x(nT)\left(\delta(t-nT) * h_o(t)\right)$$

$$= x_p(t) * h_o(t)$$

- To view the spectra (for retrieval), take CTFT

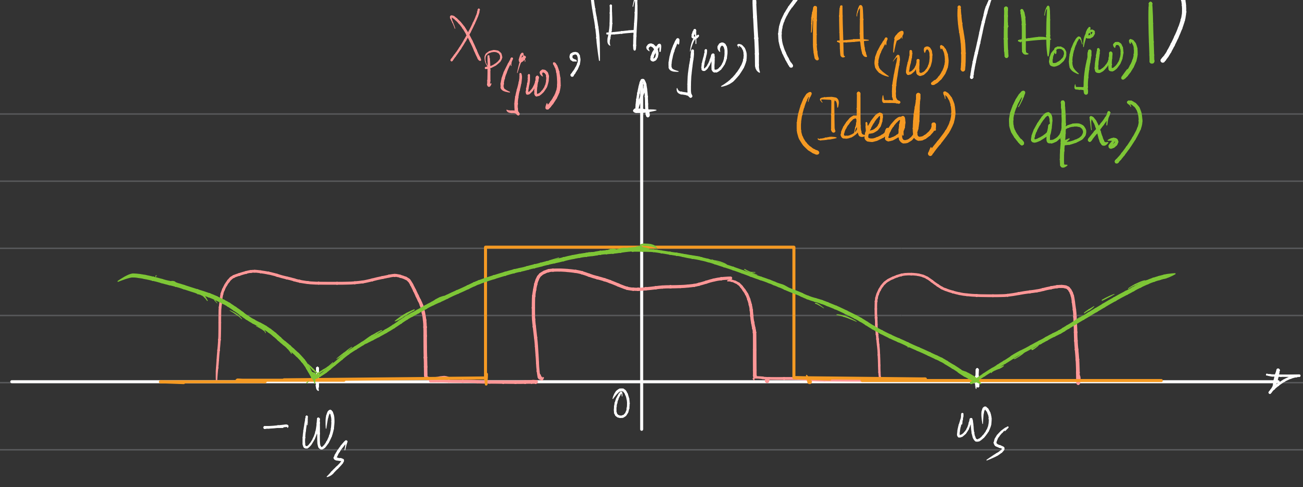
$$X_r(j\omega) = X_p(j\omega)\, H_o(j\omega)$$

where

$$H_o(j\omega) = e^{-j\omega T/2}\left(\frac{2\sin(\omega T/2)}{\omega}\right)$$

Well known shifted pulse spectra. (shifted by $T/2$)

$$|H_o(j\omega)| = \left|\frac{2\sin(\omega T/2)}{\omega}\right|$$



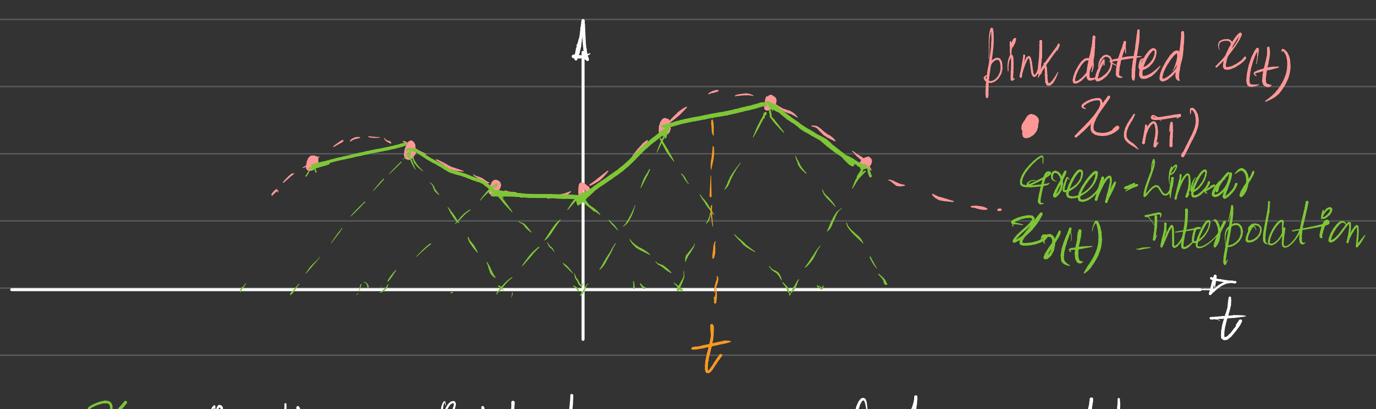
- Comparing $X_{r(j\omega)}$ with $X_{(j\omega)}$ (spectra of the original sig.)

With $H_{(j\omega)}$ With $H_{o(j\omega)}$

- In case of $H_{o(j\omega)}$ (apx.), there is distortion in the main lobe.
- Also, the side lobes pick up higher freq. noise.

- If the interpolation provided by the zero-order hold is insufficient, we can use a variety of smoother interpolation strategies.
- In particular, the zero-order hold produces an output that is discontinuous (Fig.1)
- In contrast, linear interpolation yields reconstructions that are continuous as described below.

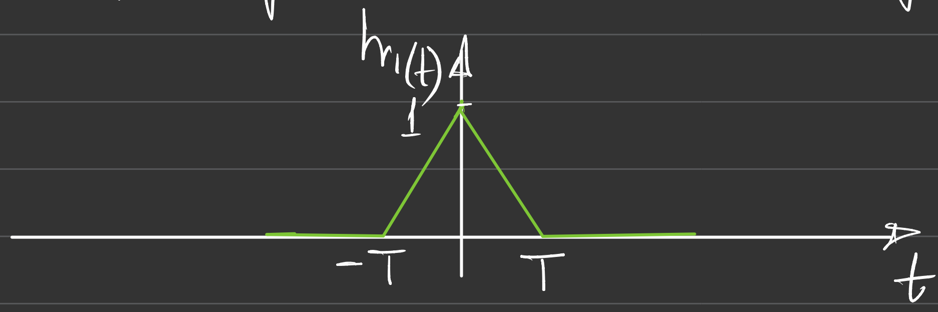
- Linear Interpolation / Interpolation with First-order Hold



- $x_{r(t)}$ is the weighted average of two samples.

$$x_r(t) = \frac{(k+1)T - t}{T} x(kT) + \frac{(t-kT)}{T} x((k+1)T)$$
$$, \forall\, kT \leq t < (k+1)T$$

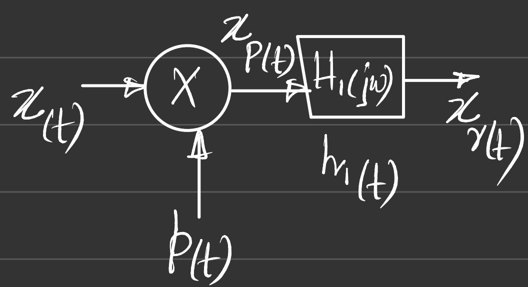
- The corresponding interpolation function is given as:



- $x_r(t)$ can be represented in terms

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\, h_1(t-nT)$$

$$= x_p(t) * h_1(t)$$

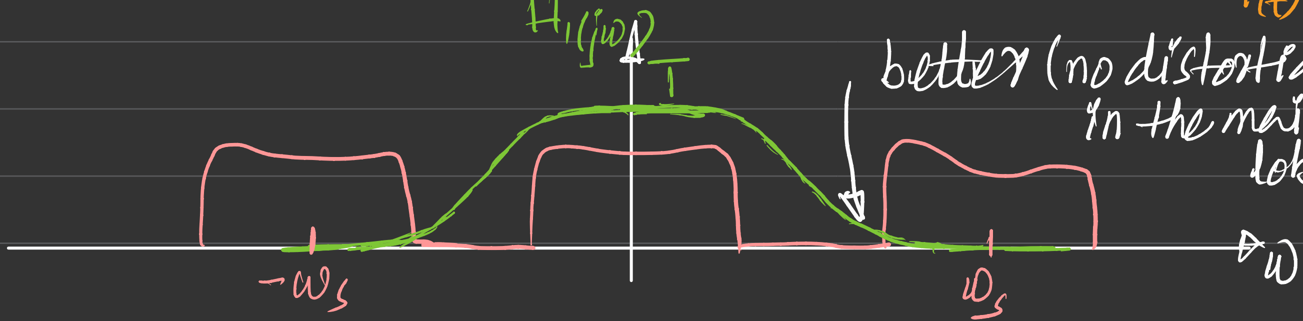


- For the spectra, take CTFT

$$X_r(j\omega) = X_p(j\omega)\, H_1(j\omega)$$

$$H_1(j\omega) = \frac{1}{T}\left(\frac{2\sin(\omega T/2)}{\omega}\right)^2$$

✓ easy to prove for $h_1(t)$



better (no distortion in the main lobe)

- In an analogous fashion, we can define second & higher order filters to produce reconstructions with higher degree of smoothness.